it is absolutely impossible to express, by means of a formula involving only rational operations and root extractons, the roots of an equation of degree higher than fourth, when the coefficients are left arbitrary. The proof of this impossibility belongs to higher parts of algebra and cannot be attempted in this course. As to the algebraic solution of cubic and biquadratic equations, the theory is comparatively simple and will be explained in this chapter and again resumed from a higher point of view in Chap. XII.

2. Cardan's Formulas. There is no loss of generality in taking the general cubic equation in the form

$$f(x) = x^3 + ax^2 + bx + c = 0$$

since division by the coefficient of x^3 does not modify the roots of the equation. By introduction of a new unknown this equation can be simplified, moreover, so that it will not contain the second power of the unknown. To this end we set

$$x = y + k$$

with k still arbitrary. By Taylor's formula

$$f(y+k) = f(k) + f'(k)y + \frac{f''(k)}{2}y^2 + \frac{f'''(k)}{6}y^3$$

and

$$f(k) = k^3 + ak^2 + bk + c, \qquad f'(k) = 3k^2 + 2ak + b,$$
$$\tfrac{1}{2}f''(k) = 3k + a, \qquad \tfrac{1}{6}f'''(k) = 1.$$

To get rid of the term involving y^2 it suffices to choose k so that

$$3k + a = 0 \qquad \text{or} \qquad k = -\frac{a}{3}.$$

Since then

$$f'\left(-\frac{a}{3}\right) = b - \frac{a^2}{3}, \qquad f\left(-\frac{a}{3}\right) = c - \frac{ba}{3} + \frac{2a^3}{27},$$

it follows that, after substituting

$$x = y - \frac{a}{3},$$

the proposed equation is transformed into

$$y^3 + py + q = 0 \tag{1}$$

where

$$p = b - \frac{a^2}{3}, \qquad q = c - \frac{ba}{3} + \frac{2a^3}{27}.$$

A cubic equation of the form (1) can be solved by means of the following device: We seek to satisfy it by setting

$$y = u + v,$$

thus introducing two unknowns u and v. On substituting this expression into (1) and arranging terms in a proper way, u and v have to satisfy the equation

$$u^3 + v^3 + (p + 3uv)(u + v) + q = 0 \tag{2}$$

with two unknowns. This problem is indeterminate unless another relation between u and v is given. For this relation we take

$$3uv + p = 0$$

or

$$uv = -\frac{p}{3}.$$

Then, it follows from (2) that

$$u^3 + v^3 = -q,$$

so that the solution of the cubic (1) can be obtained by solving the system of two equations

$$u^3 + v^3 = -q, \qquad uv = -\frac{p}{3}. \tag{3}$$

Taking the cube of the latter equation, we have

$$u^3v^3 = -\frac{p^3}{27} \tag{4}$$

and so, from equations (3) and (4), we know the sum and the product of the two unknown quantities u^3 and v^3. These quantities are the roots of the quadratic equation

$$t^2 + qt - \frac{p^3}{27} = 0.$$

Denoting them by A and B, we have then

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}},$$

$$B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}},$$

where we are at liberty to choose the square root as we please. Now owing to the symmetry between the terms u^3 and v^3 in the system (3) we can set

$$u^3 = A, \qquad v^3 = B.$$

If some determined value of the cube root of A is denoted by $\sqrt[3]{A}$, the three possible values of u will be

$$u = \sqrt[3]{A}, \qquad u = \omega\sqrt[3]{A}, \qquad u = \omega^2\sqrt[3]{A}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2}$$

is an imaginary cube root of unity. As to v it will have also three values:

$$v = \sqrt[3]{B}, \qquad v = \omega\sqrt[3]{B}, \qquad v = \omega^2\sqrt[3]{B}$$

but not every one of them can be associated with the three possible values of u, since u and v must satisfy the relation

$$uv = -\frac{p}{3}.$$

If $\sqrt[3]{B}$ stands for that cube root of B which satisfies the relation

$$\sqrt[3]{A} \cdot \sqrt[3]{B} = -\frac{p}{3},$$

then, the values of v that can be associated with

$$u = \sqrt[3]{A}, \qquad u = \omega\sqrt[3]{A}, \qquad u = \omega^2\sqrt[3]{A}$$

will be

$$v = \sqrt[3]{B}, \qquad v = \omega^2\sqrt[3]{B}, \qquad v = \omega\sqrt[3]{B}.$$

Hence, equation (1) will have the following roots:

$$\begin{aligned} y_1 &= \sqrt[3]{A} + \sqrt[3]{B}, \\ y_2 &= \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B}, \\ y_3 &= \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}. \end{aligned}$$

These formulas are known as Cardan's formulas after the name of the Italian algebraist Cardan (1501–1576), who was the first to publish them. It must be remembered that $\sqrt[3]{A}$ can be taken arbitrarily among the three possible cube roots of A, but $\sqrt[3]{B}$ must be so chosen that

$$\sqrt[3]{A} \cdot \sqrt[3]{B} = -\frac{p}{3}.$$

3. Discussion of Solution. In discussing Cardan's formulas we shall suppose that p and q are real numbers. Then, the nature of the roots will be shown to depend on the function

$$\Delta = 4p^3 + 27q^2$$

Obviously, Δ will be positive, zero, or negative. Supposing first Δ to be positive, the square root

$$\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}=\sqrt{\frac{\Delta}{108}}$$

will be real, and we shall take it positively. Then, A and B will be real, and by $\sqrt[3]{A}$ we shall mean the real cube root of A. Since p is real and

$$\sqrt[3]{A}\cdot\sqrt[3]{B}=-\frac{p}{3},$$

$\sqrt[3]{B}$ will be the real cube root of B. Hence, equation (1) has a real root

$$y_1=\sqrt[3]{A}+\sqrt[3]{B},$$

but the other two roots,

$$y_2=\omega\sqrt[3]{A}+\omega^2\sqrt[3]{B}=-\frac{\sqrt[3]{A}+\sqrt[3]{B}}{2}+i\sqrt{3}\frac{\sqrt[3]{A}-\sqrt[3]{B}}{2},$$

$$y_3=\omega^2\sqrt[3]{A}+\omega\sqrt[3]{B}=-\frac{\sqrt[3]{A}+\sqrt[3]{B}}{2}-i\sqrt{3}\frac{\sqrt[3]{A}-\sqrt[3]{B}}{2},$$

will be imaginary conjugates since A and B are not equal and consequently

$$\sqrt[3]{A}-\sqrt[3]{B}\neq 0.$$

Example 1. Let the proposed cubic equation be

$$x^3+x^2-2=0.$$

First, it must be transformed by the substitution

$$x=y-\tfrac{1}{3}.$$

The resulting equation for y (best found by synthetic division) is

$$y^3-\tfrac{1}{3}y-\tfrac{52}{27}=0,$$

so that

$$p=-\frac{1}{3},\qquad q=-\frac{52}{27},\qquad \Delta=\frac{52^2}{27}-\frac{4}{27}=100.$$

Hence,

$$\sqrt{\frac{\Delta}{108}}=\frac{5}{\sqrt{27}},\qquad A=\frac{26}{27}+\frac{5\sqrt{27}}{27},\qquad B=\frac{26}{27}-\frac{5\sqrt{27}}{27},$$

$$\sqrt[3]{A}=\tfrac{1}{3}\sqrt[3]{26+15\sqrt{3}},\quad \sqrt[3]{B}=\tfrac{1}{3}\sqrt[3]{26-15\sqrt{3}},$$

and

$$y_1=\frac{1}{3}\left(\sqrt[3]{26+15\sqrt{3}}+\sqrt[3]{26-15\sqrt{3}}\right),$$

$$y_2=-\frac{1}{6}\left(\sqrt[3]{26+15\sqrt{3}}+\sqrt[3]{26-15\sqrt{3}}\right)+\frac{i\sqrt{3}}{6}\left(\sqrt[3]{26+15\sqrt{3}}-\sqrt[3]{26-15\sqrt{3}}\right),$$

$$y_3=-\frac{1}{6}\left(\sqrt[3]{26+15\sqrt{3}}+\sqrt[3]{26-15\sqrt{3}}\right)-\frac{i\sqrt{3}}{6}\left(\sqrt[3]{26+15\sqrt{3}}-\sqrt[3]{26-15\sqrt{3}}\right).$$

Correspondingly, the roots of the proposed equation are

$$x_1 = \frac{1}{3}\left(\sqrt[3]{26+15\sqrt{3}} + \sqrt[3]{26-15\sqrt{3}} - 1\right),$$

$$x_2 = -\frac{1}{6}\left(\sqrt[3]{26+15\sqrt{3}} + \sqrt[3]{26-15\sqrt{3}} + 2\right) + \frac{i\sqrt{3}}{6}\left(\sqrt[3]{26+15\sqrt{3}} - \sqrt[3]{26-15\sqrt{3}}\right),$$

$$x_3 = -\frac{1}{6}\left(\sqrt[3]{26+15\sqrt{3}} + \sqrt[3]{26-15\sqrt{3}} + 2\right) - \frac{i\sqrt{3}}{6}\left(\sqrt[3]{26+15\sqrt{3}} - \sqrt[3]{26-15\sqrt{3}}\right).$$

The equation

$$x^3 + x^2 - 2 = 0$$

has, however, an integral root 1 and the remaining two roots,

$$-1 \pm i,$$

are imaginary.

On comparison with the expressions obtained from Cardan's formulas we discover the rather curious fact that

$$\sqrt[3]{26+15\sqrt{3}} + \sqrt[3]{26-15\sqrt{3}} = 4.$$

although both cube roots are irrational numbers. The explanation of this follows from a comparison of the imaginary roots. This comparison gives for the difference of the same cube roots

$$\sqrt[3]{26+15\sqrt{3}} - \sqrt[3]{26-15\sqrt{3}} = 2\sqrt{3},$$

whence

$$\sqrt[3]{26+15\sqrt{3}} = 2 + \sqrt{3}, \qquad \sqrt[3]{26-15\sqrt{3}} = 2 - \sqrt{3}.$$

It follows that $26 + 15\sqrt{3}$ and $26 - 15\sqrt{3}$ are cubes of numbers $2 + \sqrt{3}$ and $2 - \sqrt{3}$. Such simplification of cube roots occurs always when the cubic equation has a rational root but not otherwise.

Example 2. To solve the equation

$$x^3 + 9x - 2 = 0.$$

Here the preliminary transformation is not necessary and Cardan's formulas can be applied directly. We have

$$p = 9, \qquad q = -2, \qquad \Delta = 3024, \qquad \frac{\Delta}{108} = 28,$$

$$A = 1 + \sqrt{28}, \qquad B = 1 - \sqrt{28}.$$

Consequently, the real root is

$$\sqrt[3]{\sqrt{28}+1} - \sqrt[3]{\sqrt{28}-1}$$

while the imaginary roots are

$$-\frac{1}{2}\left(\sqrt[3]{\sqrt{28}+1} - \sqrt[3]{\sqrt{28}-1}\right) \pm \frac{i\sqrt{3}}{2}\left(\sqrt[3]{\sqrt{28}+1} + \sqrt[3]{\sqrt{28}-1}\right).$$

To calculate these roots approximately, use can be made of any of the well-known handbooks that contain tables of squares, cubes, square roots, and cube roots. From such tables it is found that

$$\sqrt{28} + 1 = 6.2915026, \qquad \sqrt{28} - 1 = 4.2915026,$$

and further

$$\sqrt[3]{\sqrt{28} + 1} = 1.8460840, \qquad \sqrt[3]{\sqrt{28} - 1} = 1.6250615.$$

Hence, the real root of the proposed equation is approximately

$$1.8460840 - 1.6250615 = 0.2210225$$

with the degree of approximation allowed by seven-place tables.

In case $\Delta = 0$

$$A = B = -\frac{q}{2},$$

and the roots of the equation

$$y^3 + py + q = 0$$

are

$$y = 2\sqrt[3]{-\frac{q}{2}}, \qquad y_2 = \sqrt[3]{\frac{q}{2}}, \qquad y_3 = \sqrt[3]{\frac{q}{2}}.$$

Thus, $y_2 = y_3$ is a double root unless $q = 0$, which implies $p = 0$, when all three roots are equal to 0 and the equation

$$y^3 = 0$$

is trivial.

Problems

Solve the cubic equations:

1. $x^3 - 6x - 6 = 0.$
2. $x^3 - 12x - 34 = 0.$
3. $x^3 + 9x - 6 = 0.$
4. $x^3 + 18x - 6 = 0.$
5. $2x^3 + 6x + 3 = 0.$
6. $2x^3 - 3x + 5 = 0.$
7. $3x^3 - 6x^2 - 2 = 0.$
8. $x^3 + 6x^2 - 36 = 0.$
9. $x^3 + 3x^2 + 9x + 14 = 0.$
10. $x^3 + 6x^2 + 6x + 5 = 0.$
11. $x^3 - 15x^2 + 105x - 245 = 0.$
12. $8x^3 + 12x^2 + 102x - 47 = 0.$
13. $x^3 - 2x + 2 = 0.$
14. $x^3 + 3x - 2 = 0.$
15. $x^3 + 6x^2 + 9x + 8 = 0.$
16. $8x^3 + 12x^2 + 30x - 3 = 0.$

Show that

17. $\sqrt[3]{\sqrt{5} + 2} - \sqrt[3]{\sqrt{5} - 2} = 1.$
18. $\sqrt[3]{7 + \sqrt{50}} + \sqrt[3]{7 - \sqrt{50}} = 2.$
19. $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10} = 2.$
20. $\sqrt[3]{\sqrt{243} + \sqrt{242}} - \sqrt[3]{\sqrt{243} - \sqrt{242}} = 2\sqrt{2}.$

21. What is the outer radius of a spherical shell of thickness 1 in. if the volume of the shell is equal to the volume of the hollow space inside?

22. Solve Prob. 21 if the volume of the hollow space is twice the volume of the shell.

23. An open box has the shape of a cube with each edge 10 in. long. If the capacity of the box is 500 cu. in., what is the thickness of the walls? The walls are supposed to be uniformly thick.

4. Irreducible Case. We return now to the discussion of the general solution and consider what happens when $\Delta < 0$. A curious phenomenon occurs here, for in this case

$$\sqrt{\frac{q^2}{4}+\frac{p^3}{27}} = i\sqrt{\frac{-\Delta}{108}}$$

is purely imaginary and both numbers

$$A = -\frac{q}{2}+i\sqrt{\frac{-\Delta}{108}}, \qquad B = -\frac{q}{2}-i\sqrt{\frac{-\Delta}{108}}$$

are imaginary, so that the roots of equation (1) in Sec. 2 are expressed through the cube roots of imaginary numbers, and yet all three of them are real. To see this let

$$\sqrt[3]{A} = a + bi$$

be one of the cube roots of A. Since B is conjugate to A, the number $a - bi$ will be one of the cube roots of B, and it must be taken equal to $\sqrt[3]{B}$ in order to satisfy the condition

$$\sqrt[3]{A}\cdot\sqrt[3]{B} = -\frac{p}{3}.$$

Thus,

$$\sqrt[3]{A} = a + bi, \qquad \sqrt[3]{B} = a - bi,$$

and from Cardan's formulas it follows that the roots

$$\begin{aligned} y_1 &= 2a, \\ y_2 &= (a+bi)\omega + (a-bi)\omega^2 = -a - b\sqrt{3}, \\ y_3 &= (a+bi)\omega^2 + (a-bi)\omega = -a + b\sqrt{3} \end{aligned}$$

are real and, moreover, unequal. It is clear that $y_2 \neq y_3$. If $y_1 = y_2$, we should have

$$b = -a\sqrt{3},$$

so that

$$\sqrt[3]{A} = a(1 - i\sqrt{3}).$$

But then

$$A = a^3(1 - i\sqrt{3})^3 = -8a^3$$

would be real, which is not true. Similarly, it is shown that $y_1 \neq y_3$.

Example. To solve the equation

$$y^3 - 3y + 1 = 0.$$

In this case

$$p = -3, \qquad q = 1, \qquad \Delta = -81, \qquad \sqrt{\frac{-\Delta}{108}} = \frac{\sqrt{3}}{2},$$

$$A = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \omega, \qquad B = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \omega^2.$$

and the real roots are presented in the form

$$y_1 = \sqrt[3]{\omega} + \sqrt[3]{\omega^2},$$
$$y_2 = \omega\sqrt[3]{\omega} + \omega^2\sqrt[3]{\omega^2},$$
$$y_3 = \omega^2\sqrt[3]{\omega} + \omega\sqrt[3]{\omega^2}.$$

These expressions are not suitable for direct calculation because of the cube roots of imaginary numbers. If we try to find

$$\sqrt[3]{\omega} = a + bi$$

algebraically, we are led to the solution of the two simultaneous equations

$$a^3 - 3ab^2 = -\frac{1}{2}, \qquad 3a^2b - b^3 = \frac{\sqrt{3}}{2}.$$

Solving for b^2 in the first and substituting

$$b^2 = \frac{2a^3 + 1}{6a}$$

into the second, we find

$$b\left(3a^2 - \frac{2a^3 + 1}{6a}\right) = \frac{\sqrt{3}}{2},$$

whence

$$b = \frac{3\sqrt{3}a}{16a^3 - 1}, \qquad b^2 = \frac{27a^2}{(16a^3 - 1)^2}.$$

Equating the two expressions for b^2, we have the equation

$$\frac{2a^3 + 1}{6a} = \frac{27a^2}{(16a^3 - 1)^2},$$

which after due simplifications becomes

$$(2a)^9 + 3(2a)^6 - 24(2a)^3 + 1 = 0.$$

Setting $x = 8a^3$, we have for x a cubic equation

$$x^3 + 3x^2 - 24x + 1 = 0.$$

which by the substitution $x = y - 1$ is transformed into

$$y^3 - 27y - 27 = 0;$$

or, setting $y = -3z$, into

$$z^3 - 3z + 1 = 0.$$

But this is the same equation that we wanted to solve. Consequently, we did not advance a step in trying to find a and b by an algebraic process. The fact that the real roots of a cubic equation

$$y^3 + py + q = 0$$

in case

$$4p^3 + 27q^2 < 0$$

are presented in a form involving the cube roots of imaginary numbers puzzled the old algebraists for a long time, and this case was called by them *casus irreducibilis*, irreducible case. We know now that, for instance, when p and q are rational numbers, but among the three real roots of an equation

$$y^3 + py + q = 0$$

none is rational, it is absolutely impossible to express any of these roots in a form involving only real radicals of any kind.

5. Trigonometric Solution. In spite of the algebraic difficulties inherent in the irreducible case, it is possible to present the roots in a form suitable for numerical computation by extracting the cube root of

$$A = -\frac{q}{2} + i\sqrt{-\frac{q^2}{4} - \frac{p^3}{27}}$$

trigonometrically. The square of the modulus of A is

$$\rho^2 = \left(-\frac{q}{2}\right)^2 - \left(\frac{q}{2}\right)^2 - \frac{p^3}{27} = \left(\frac{-p}{3}\right)^3$$

whence

$$\rho = \left(\frac{-p}{3}\right)^{3/2} = \frac{-p\sqrt{-p}}{\sqrt{27}}.$$

The argument of A can be determined either by its cosine

$$\cos\phi = \frac{\sqrt{27}q}{2p\sqrt{-p}}$$

or by its tangent

$$\tan\phi = \frac{-\sqrt{-\Delta}}{q\sqrt{27}}$$

on the condition that ϕ is taken in the first or second quadrant according as q is negative or positive. Having found ρ and ϕ, we can take

$$\sqrt[3]{A} = \sqrt[3]{\rho}\left(\cos\frac{\phi}{3} + i\sin\frac{\phi}{3}\right) = \sqrt{\frac{-p}{3}}\left(\cos\frac{\phi}{3} + i\sin\frac{\phi}{3}\right),$$

and

$$\sqrt[3]{B} = \sqrt{\frac{-p}{3}}\left(\cos\frac{\phi}{3} - i\sin\frac{\phi}{3}\right).$$

Then, since

$$\omega = \cos 120° + i\sin 120°,$$

the roots y_1, y_2, y_3 will be given by

$$y_1 = 2\sqrt{\frac{-p}{3}}\cos\frac{\phi}{3},$$

$$y_2 = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\phi}{3} + 120°\right),$$

$$y_3 = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\phi}{3} + 240°\right).$$

In practice it is more convenient to present y_2 and y_3 thus:

$$y_2 = -2\sqrt{\frac{-p}{3}}\cos\left(60^\circ - \frac{\phi}{3}\right),$$

$$y_3 = -2\sqrt{\frac{-p}{3}}\cos\left(60^\circ + \frac{\phi}{3}\right).$$

Example 1. For the equation

$$y^3 - 3y + 1 = 0$$

we have

$$\sqrt{\frac{-p}{3}} = 1, \qquad \cos\phi = -\frac{1}{2};$$

hence, $\phi = 120^\circ$ and

$$y_1 = 2\cos 40^\circ, \qquad y_2 = -2\cos 20^\circ, \qquad y_3 = 2\cos 80^\circ.$$

Approximate values of the roots can be found directly from the trigonometric tables and are

$$\begin{aligned} y_1 &= 1.5320888862, \\ y_2 &= -1.8793852416, \\ y_3 &= 0.3472963554. \end{aligned}$$

Example 2. To solve the equation

$$y^3 - 7y - 7 = 0.$$

For this equation

$$p = -7, \qquad q = -7, \qquad -\Delta = 49$$

and

$$\tan\phi = \frac{1}{\sqrt{27}}.$$

Further computation was made with six-place tables of logarithms. The calculations and results can be presented as follows:

$$\begin{aligned} \log 27 &= 1.431364 \\ \log\sqrt{27} &= 0.715682 \\ \log\tan\phi &= 9.284318 \\ \phi &= 10^\circ 53' 36''.2 \\ \tfrac{1}{3}\phi &= 3^\circ 37' 52''.0 \end{aligned}$$

$$\begin{aligned} \log 7 &= 0.845098 \\ \log 3 &= 0.477121 \\ \log \tfrac{7}{3} &= 0.367977 \\ \log\sqrt{\tfrac{7}{3}} &= 0.183988 \\ \log 2 &= 0.301030 \\ \log 2\sqrt{\tfrac{7}{3}} &= 0.485018 \end{aligned}$$

$$\begin{aligned} \log\cos\frac{\phi}{3} &= 9.999128 - 10 \\ & \ 0.485018 \\ \log y_1 &= 0.484146 \\ y_1 &= 3.04892 \\ \log\cos\left(60^\circ + \frac{\phi}{3}\right) &= 9.647528 - 10 \\ & \ 0.485018 \\ \log(-y_2) &= 0.132546 \\ -y_2 &= 1.35689 \\ \log\cos\left(60^\circ - \frac{\phi}{3}\right) &= 9.743387 - 10 \\ & \ 0.485018 \\ \log(-y_3) &= 0.228405 \\ -y_3 &= 1.69202 \end{aligned}$$

$$\begin{aligned} y_1 &= 3.04892 \\ y_2 &= -1.35689 \\ y_3 &= -1.69202 \\ \hline & 0.00001 \end{aligned}$$